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GRADIENT ELASTICITY WITH SURFACE ENERGY: MODE-III CRACK PROBLEM

I. VARDOULAKIS
Department of Engineering Science, National Technical University of Athens, GR-15773, Athens, Greece

G. EXADAKTYLOS
Department of Mineral Resources Engineering, Technical University of Crete, GR-73100, Hania, Greece

and

E. AIFANTIS
Center of Mechanics of Materials and Instabilities, Michigan Technological University, MI-49931, Houghton, U.S.A. and Laboratory of Mechanics and Materials, Aristotle University of Thessaloniki, GR-54006, Thessaloniki, Greece

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Abstract—In this paper an anisotropic strain-gradient dependent theory of elasticity is exploited, which contains both volumetric and surface energy gradient dependent terms. The theory is applied to the solution of the mode-III crack problem and is extending previous results by Aifantis and co-workers. The two boundary value problems corresponding to the "unclamped" and "clamped" crack tips, respectively, are solved analytically. It turns out that the first problem is physically questionable for some values of the surface energy parameter, whereas the second boundary value problem is leading to a cusping crack, which is consistent with Barenblatt's theory without the incorporation of artificial assumptions. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

Standard continuum-mechanics theories cannot describe situations dominated by microstructural effects, e.g. load or geometrically induced stress singularities, since the influence of these effects is not properly accounted by the standard continuum models. Generalized continuum theories, on the other hand, such as higher-order gradient theories or couple-stress theories can often describe adequately the effect of microstructure on the macroscale through the solution of properly formulated boundary value problems. It should be noticed that for the corresponding generalized continuum theories with a variety of names and microstructures of various degrees of rigor and complexity have been invented; e.g. Cosserat continua or micropolar media, oriented media, continuum theories with directors, multipolar continua, multistructured or micromorphic continua, non-local media and others (cf. Hermann, 1972). The state-of-the-art of this evolution in the mid 1960s was reflected in the collection of papers presented at the historical IUTAM Symposium on the "Mechanics of Generalized Continua", in Freudenstadt and Stuttgart in 1967 (Kroner, 1967). Newly, the interest to such theories is rekindled through the idea of connecting micromechanics with fracture and failure of solids (see for example, Muhlhaus and Vardoulakis, 1987, and for extensive literature review in Vardoulakis and Sulem, 1995, chapters 8 and 9).

The isotropic, higher-gradient, linear elasticity theory was developed essentially by Mindlin (1964; 1965a). Mindlin's (1965a) theory has been recently explored as far as its mathematical potential is concerned in a comprehensive paper by Wu (1992). In this paper Wu states (p. 101), that "... since it is not possible to include a term linear in [the strain gradient] E_{ijk} in [the strain energy density] W, it is not possible to introduce a material constant to induce a self-equilibrating state for defining, say, surface free energy ...". This is clear if one considers that the strain gradient is a third order tensor, and, in order to include its effect in the strain energy density function, one needs a director whose existence is excluded from the isotropy assumption. However, if one restricts the effect of gradient of strain to its trace (i.e. to the gradient of the strain dilatation), one arrives to Mindlin's (1965a) surface strain energy density term that accounts for the Laplacian of the dilatation. It should be pointed out, however, that in some boundary-value problems, like the anti-plane shear problem, such a constitutive theory, due to the absence of dilatation, degenerates, and the effect of the dilatation-gradient term vanishes.

The present paper can be seen more as a further investigation of the questions posed earlier by Aifantis and co-workers (1992a; 1992b; 1993; 1994). There the potential of applying gradient elasticity theory to linear elastic fracture mechanics is explored on the basis of its applicability into describing the stress and displacement fields in the vicinity of Griffith cracks. For simplicity reasons Aifantis and co-workers have studied only the effect of the volume strain–gradient energy term, and, as he states, his theory may be viewed as a particular case of Mindlin's (1965a) original theory, involving only one material constant. The present paper deviates from this path suggesting a constitutive model that accounts for surface strain–gradient energy terms which persist in anti-plane shear. The background for this constitutive model is sketched briefly below.

At practically the same time as publication of the pioneering papers by Mindlin (1964; 1965a) and Mindlin and Eshel (1968), Professor Germain has encouraged the communication to the French Academy of Sciences of the ideas of Casal (1961; 1963; 1972), which in turn seem to have inspired Germain's (1973a; 1973b) fundamental papers on the continuum mechanics structure of the grade-2 or higher grade theories. In our paper, we want to give full credit to Casal's original idea, who was first to see the connection between surface tension effects and the anisotropic gradient elasticity theory. For this reason we rework here Casal's original idea on the 1D tension bar problem and provide the simplest possible generalization of the corresponding constitutive theory that accounts for only two material constants having the dimension of length: one, say ℓ, responsible for volumetric strain–gradient terms, and another, ℓ', responsible for surface strain–gradient terms. Of course such ideas are amenable to generalizations of various degrees of complexity. However, one should keep in mind that already the determination of the two material lengths ℓ and ℓ' constitutes a formidable experimental challenge, and as we will show here this might be possible only through precise monitoring of the crack profile.

Here we derive for two specific boundary value problems of mode-III crack deformation (or tearing mode) exact solutions in integral form. These are: (a) the problem where no restriction on the displacements at the crack tip is posed (traction boundary value problem), and (b) the problem where the tips of the crack are clamped (mixed boundary value problem). The results are compared with those of Altan and Aifantis (1992b) and Ru and Aifantis (1993). In both problems the stress at the crack tip exhibits the inverse square root singularity as in the classical Linear Elastic Fracture Mechanics (LEFM) theory, however, the strain field remains finite everywhere.

It turns out that if the surface energy parameter is included in the analysis of the first problem and for sufficiently large values of this parameter, the crack faces interpenetrate each other. This means that the specific boundary value problem is improperly posed at least for these values of the surface energy parameters which give considerable values of negative anti-plane shear displacement as compared to the length of the crack. The solution of the second problem yields a crack shape forming cusps of the first kind at the crack tip, which is consistent with Barenblatt's (1962) "cohesive-zone" theory, but without requiring an extra assumption on the existence and effect of interatomic forces beyond those implied by the gradient elasticity. It was also found that the incorporation of volumetric strain gradient energy terms leads in general into a crack stiffening effect. In this paper we will show that surface strain gradient terms may have the opposite effect which in general is a welcome property of a mathematical model as far as description of experimental data is concerned.

2. ANISOTROPIC GRADIENT ELASTICITY THEORY WITH SURFACE ENERGY

An early formulation of a simple linear continuum theory with microstructure can be found in a rather unnoticed publication by Casal (1961), quoted in the papers by Germain

(1973a; 1973b). It is noted that Casal's model cannot be directly embedded in Mindlin's (1964) linear, isotropic elasticity theory with microstructure, because the former is an anisotropic elasticity model. In Appendix A Casal's model is demonstrated in the 1D case of a tension bar. The 3D generalization of Casal's gradient dependent anisotropic elasticity with surface energy is straightforward leading to the following expression for the strain energy density function

$$w = \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + G\varepsilon_{ij}\varepsilon_{ji} + G\ell^2\partial_k\varepsilon_{ij}\partial_k\varepsilon_{ji} + G\ell_k\partial_k(\varepsilon_{ij}\varepsilon_{ji}) \tag{1}$$

where $\partial_k \equiv \partial/\partial x_k$, λ and G are Lame's constants, ℓ, ℓ' are characteristic lengths of the material defined previously,

$$\varepsilon_{ij} = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{2}$$

are the strains, u_i the displacements, and

$$\ell_k = \ell' \nu_k, \quad \nu_k\nu_k = 1 \tag{3}$$

is a director. Accordingly, eqn (1) defines a gradient anisotropic elasticity with constant characteristic directors ℓ_k. The last term in eqn (1) has the meaning of surface-energy since

$$\int\partial_r(\ell_r\varepsilon_{pq}\varepsilon_{qp})\,\mathrm{d}V = \ell'\int(\varepsilon_{pq}\varepsilon_{qp})(\nu_r n_r)\,\mathrm{d}S \tag{4}$$

and n_r is the outward unit normal to the boundary. Here, the particular case is considered where $\nu_r \equiv n_r$, which physically corresponds to surface-parallel micro-cracks.

In this case, from eqn (1) we obtain the following constitutive relations for the stresses and double stresses

$$\left.\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2G(\varepsilon_{ij} - \ell^2\nabla^2\varepsilon_{ij}) \\ \tau_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} + 2G\ell_k\partial_j\varepsilon_{ij} \\ \mu_{kij} &= 2G\ell_k\varepsilon_{ij} + 2G\ell^2\partial_k\varepsilon_{ij} \end{aligned}\right\} \tag{5}$$

where δ_{ij} is Kronecker delta. These stresses obey the following equilibrium conditions in the volume V in the case of absence of body forces (Mindlin, 1964)

$$\partial_i\sigma_{ij} = 0, \tag{6}$$

as well as, the following set of static boundary conditions on the surface S

$$n_j\tau_{jk} - n_i n_j D\mu_{ijk} - (n_i D_j + n_j D_i)\mu_{ijk} + (n_i n_j D_\ell n_\ell - D_j n_i)\mu_{ijk} = \tilde{P}_k \tag{7}$$

$$n_i n_j \mu_{ijk} = \tilde{R}_k, \tag{8}$$

where $\tilde{P}_k$, $\tilde{R}_k$ are the specified tractions and double tractions, respectively, on the surface S, and

$$D = n_k\partial_k, \quad D_k = (\delta_{k\ell} - n_k n_\ell)\partial_\ell. \tag{9}$$

3. THE ANTI-PLANE SHEAR CRACK PROBLEM

For the half-plane $y \geqslant 0$ problem of pure shear or mode-III crack deformation (Fig. 1), we have

$$u_x = u_y = 0, \quad u_z \neq 0$$
$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0, \quad \sigma_{xz} \neq 0, \quad \sigma_{yz} \neq 0, \quad \sigma_{xy} = 0. \tag{10}$$

Thus, the only non-vanishing component of displacement is u_z which is a function of x and y. In view of the earlier derived constitutive equations of gradient elasticity with surface energy we have

$$\sigma_{xz} = 2G(\varepsilon_{xz} - \ell^2 \nabla^2 \varepsilon_{xz})$$
$$\sigma_{yz} = 2G(\varepsilon_{yz} - \ell^2 \nabla^2 \varepsilon_{yz})$$

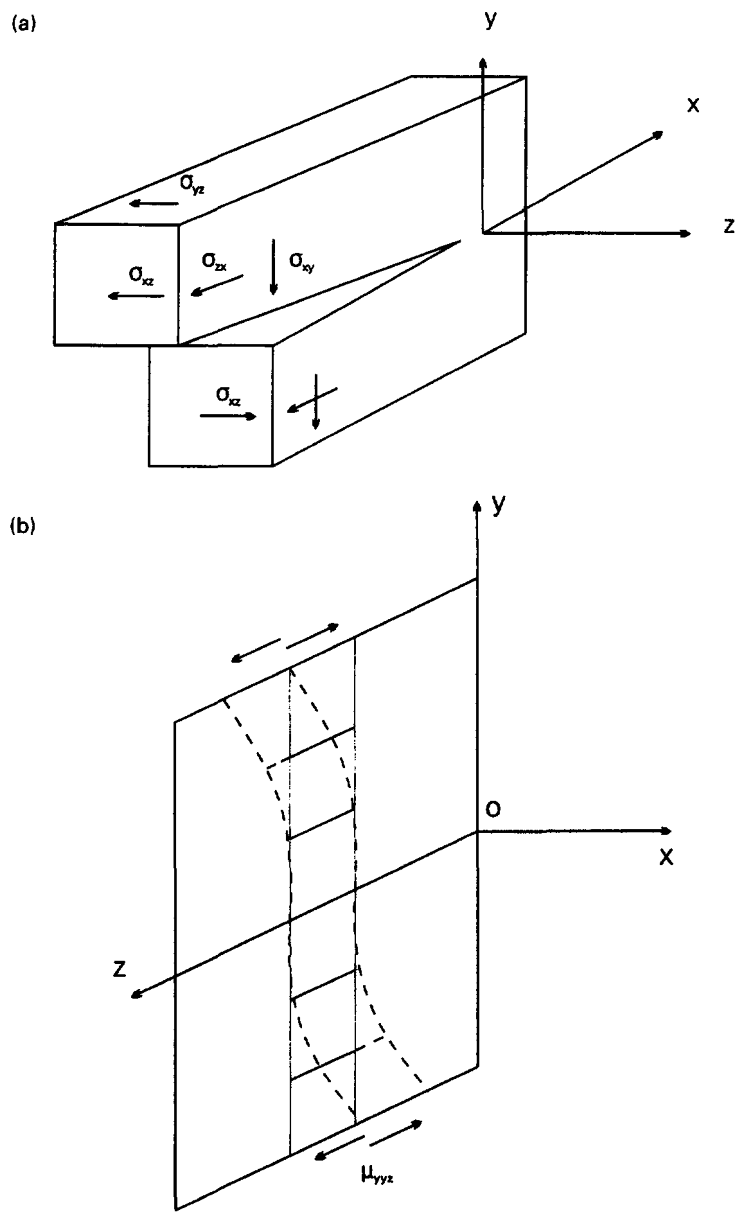

Fig. 1. (a) Crack geometry and stress notation; (b) double stress.

$$\mu_{xxz} = 2G\ell^2 \frac{\partial}{\partial x}\varepsilon_{xz}$$

$$\mu_{xyz} = 2G\ell^2 \frac{\partial}{\partial x}\varepsilon_{yz}$$

$$\mu_{yxz} = 2G\left(\ell'\varepsilon_{xz} + \ell^2 \frac{\partial}{\partial y}\varepsilon_{xz}\right)$$

$$\mu_{yyz} = 2G\left(\ell'\varepsilon_{yz} + \ell^2 \frac{\partial}{\partial y}\varepsilon_{yz}\right), \tag{11}$$

where G is the shear modulus and the strains are defined, as usual, by the relations

$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial u_z}{\partial x}, \quad \varepsilon_{yz} = \frac{1}{2}\frac{\partial u_z}{\partial y}. \tag{12}$$

All other components of the strain and stress tensors vanish and the only non-trivial equilibrium equation becomes

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0. \tag{13}$$

Substituting the expressions for the stress components as they are given by eqn (11) into eqn (13), we obtain

$$\frac{\partial}{\partial x}\left\{G\left(\frac{\partial u_z}{\partial x} - \ell^2\nabla^2\frac{\partial u_z}{\partial x}\right)\right\} + \frac{\partial}{\partial y}\left\{G\left(\frac{\partial u_z}{\partial y} - \ell^2\nabla^2\frac{\partial u_z}{\partial y}\right)\right\} = 0. \tag{14}$$

The above equation can be further simplified to yield

$$-\ell^2\nabla^4 u_z + \nabla^2 u_z = 0 \tag{15}$$

with

$$u_z(x,y) = -u_z(x,-y), \tag{16}$$

i.e. the only surviving displacement component is an odd function of y. Relationship (15) is essentially a singular perturbation of the classical theory and it is obvious that as $\ell \to 0$ it is reduced to the classical LEFM harmonic equation.

It can be shown that the general solution of the above fourth order partial differential equation admits the representation (Mindlin, 1965b)

$$u_z(x,y) = u_z^c(x,y) + u_z^g(x,y) \tag{17}$$

where u_z^c, u_z^g are real functions which are the solutions of the harmonic and Helmholtz's equations, respectively.

If the traction specified on the crack surface is an even function of x then u_z is symmetric with respect to x, i.e. $u_z(x,y) = u_z(-x,y)$. This justifies the application of the Fourier cosine transform to the harmonic equation

$$\nabla^2 u_z^c = 0 \tag{18}$$

and considering that the displacements must vanish at infinity it is found that

$$u_z^c(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty A(\xi)\,\mathrm{e}^{-y\xi}\cos x\xi\,\mathrm{d}\xi \quad y \geqslant 0. \tag{19}$$

Similarly, by applying the Fourier cosine transform to Helmholtz's equation

$$u_z^g - \ell^2\nabla^2 u_z^g = 0 \tag{20}$$

and solving the resultant ordinary differential equation of second order it is obtained

$$u_z^g(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty B(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\cos x\xi\,\mathrm{d}\xi \quad y \geqslant 0. \tag{21}$$

Combining eqns (17), (19) and (21) the following result is derived (also derived by Altan and Aifantis, 1992b)

$$u_z(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{A(\xi)\,\mathrm{e}^{-y\xi} + B(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\cos x\xi\,\mathrm{d}\xi \quad y \geqslant 0 \tag{22}$$

where $A(\xi)$, $B(\xi)$ are unknown functions to be determined from the boundary conditions of the problem. Accordingly, the expressions for the only surviving stresses σ_{xz}, σ_{yz} and the double stresses μ_{xxz}, μ_{xyz}, μ_{yxz}, μ_{yyz} can be obtained from eqns (11), (12) and (22) as follows

$$\sigma_{xz}(x,y) = -G\sqrt{\frac{2}{\pi}}\int_0^\infty \xi A(\xi)\,\mathrm{e}^{-y\xi}\sin x\xi\,\mathrm{d}\xi$$

$$\sigma_{yz}(x,y) = -G\sqrt{\frac{2}{\pi}}\int_0^\infty \xi A(\xi)\,\mathrm{e}^{-y\xi}\cos x\xi\,\mathrm{d}\xi$$

$$\mu_{xxz}(x,y) = -G\ell^2\sqrt{\frac{2}{\pi}}\int_0^\infty \xi^2\left\{A(\xi)\,\mathrm{e}^{-y\xi} + B(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\cos x\xi\,\mathrm{d}\xi$$

$$\mu_{xyz}(x,y) = G\ell^2\sqrt{\frac{2}{\pi}}\int_0^\infty \xi\left\{\xi A(\xi)\,\mathrm{e}^{-y\xi} + \sqrt{\xi^2+\frac{1}{\ell^2}}\,B(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\sin x\xi\,\mathrm{d}\xi$$

$$\mu_{yxz}(x,y) = -G\sqrt{\frac{2}{\pi}}\int_0^\infty \left\{[\ell'\xi - \ell^2\xi^2]A(\xi)\,\mathrm{e}^{-y\xi}\right.$$
$$\left. + \left[\ell'\xi - \ell^2\xi\sqrt{\xi^2+\frac{1}{\ell^2}}\right]B(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\sin x\xi\,\mathrm{d}\xi$$

$$\mu_{yyz}(x,y) = -G\sqrt{\frac{2}{\pi}}\int_0^\infty \left\{[\ell'\xi - \ell^2\xi^2]A(\xi)\,\mathrm{e}^{-y\xi}\right.$$
$$\left. + \left[\ell'\sqrt{\xi^2+\frac{1}{\ell^2}} - (1+\ell^2\xi^2)\right]B(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\cos x\xi\,\mathrm{d}\xi \quad y \geqslant 0. \tag{23}$$

Problem I: *unclamped crack tips*

Altan and Aifantis (1992b) employed a special isotropic gradient elasticity theory and demonstrated that if the load is prescribed along the surface of the physical crack (traction boundary value problem), then the displacement boundary condition (mixed boundary value problem), usually imposed outside the crack, is a consequence rather than a requirement. In the same spirit, we now state the problem in the present gradient elasticity theory of a finite straight crack in a field of constant anti-plane shear stress applied on its surface and vanishing at infinity. Let S be the complement of the line segment $-\alpha < x < \alpha$, $y = 0$ extended on the half-plane $y \geqslant 0$ with 2α denoting the length of the crack. We see the solution in S of the field eqns (23) subject to the inhomogeneous boundary conditions

$$\left.\begin{array}{ll} \sigma_{yz}(x,0) = -\sigma_0 & 0 \leqslant x < \alpha \\ \mu_{yyz}(x,0) = 0 & 0 \leqslant x < \infty \end{array}\right\}, \tag{24}$$

where σ_0 is the intensity of the applied loading $[FL^{-2}]$, and to the homogeneous regularity conditions at infinity

$$\sigma_{xz}, \sigma_{yz} \to 0, \quad \mu_{xxz}, \mu_{xyz}, \mu_{yxz}, \mu_{yyz} \to 0, \quad u_z \to 0 \quad \text{as} \sqrt{x^2+y^2} \to \infty. \tag{25}$$

The first of conditions (24) and the first and third of (25) are the classical ones, while the second one of conditions (24) is an extra boundary condition required as a result of the gradient terms.

The above boundary conditions take the following form in view of the last two equations of (23)

$$\left.\begin{array}{ll} \displaystyle\int_0^\infty \xi A(\xi) \cos x\xi \, \mathrm{d}\xi = \sqrt{\frac{\pi}{2}} \frac{\sigma_0}{G} & 0 \leqslant x < \alpha \\ \displaystyle\int_0^\infty \left\{ [\ell'\xi - \ell^2\xi^2] A(\xi) + \left[\ell' \sqrt{\xi^2 + \frac{1}{\ell^2}} - (1+\ell^2\xi^2) \right] B(\xi) \right\} \cos x\xi \, \mathrm{d}\xi = 0 & 0 \leqslant x < \infty \end{array}\right\}. \tag{26}$$

The second of relations (26) yields

$$B(\xi) = \frac{\ell^2\xi^2 - \ell'\xi}{\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2} - (1+\ell^2\xi^2)} A(\xi) \quad 0 \leqslant \xi < \infty. \tag{27}$$

Thus, if the unknown function $A(\xi)$ can be determined from the first of the integral equations (26), the problem is fully solved. This is done here by means of a semi-inverse method which is guided by the classical LEFM solution. Accordingly, the following representation $u_z^c(x,0)$ is assumed, which as seen below is justified by a direct *a posteriori* verification

$$u_z^c(x,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\xi) \cos x\xi \, \mathrm{d}\xi = C_1 \sqrt{\alpha^2 - x^2} \quad 0 \leqslant x < \alpha \tag{28}$$

where C_1 is a constant to be found from the boundary conditions. Applying the Fourier cosine inversion formula to eqn (28), we have

$$A(\xi) = \sqrt{\frac{\pi}{2}} C_1 \alpha \frac{J_1(\alpha\xi)}{\xi} \quad 0 < \xi < \infty \tag{29}$$

where J_n is the usual Bessel function of the first kind and of order n.

Introducing the above expression in the first boundary condition (26), it is found that

$$C_1 \int_0^\infty J_1(\alpha\xi)\,\mathrm{d}\xi = \frac{\sigma_0}{G\alpha} \tag{30}$$

and according to the well-known identity

$$\int_0^\infty J_1(\alpha\xi)\,\mathrm{d}\xi = \frac{1}{\alpha} \tag{31}$$

we have

$$C_1 = \frac{\sigma_0}{G}. \tag{32}$$

Finally, combination of eqns (29) and (32) yields

$$A(\xi) = \sqrt{\frac{\pi}{2}}\frac{\sigma_0\alpha}{G}\frac{J_1(\alpha\xi)}{\xi}. \tag{33}$$

It can be easily seen that the above expression for $A(\xi)$ satisfies the first of boundary conditions (26), and the regularity conditions (25), which is consistent with the assumption in (28). Therefore, in accordance to (27) and (33), the final expression for the mode-III displacement, stresses and strains take the form

$$u_z(x,y) = \frac{\sigma_0\alpha}{G}\int_0^\infty \xi^{-1}J_1(\alpha\xi)\left\{\mathrm{e}^{-y\xi} - \frac{\ell^2\xi^2-\ell'\xi}{1+\ell^2\xi^2-\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}}\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\cos x\xi\,\mathrm{d}\xi$$

$$\sigma_{xz}(x,y) = -\sigma_0\alpha\int_0^\infty J_1(\alpha\xi)\,\mathrm{e}^{-y\xi}\sin x\xi\,\mathrm{d}\xi$$

$$\sigma_{yz}(x,y) = -\sigma_0\alpha\int_0^\infty J_1(\alpha\xi)\,\mathrm{e}^{-y\xi}\cos x\xi\,\mathrm{d}\xi$$

$$\mu_{xxz}(x,y) = -\sigma_0\alpha\ell^2\int_0^\infty \xi J_1(\alpha\xi)\left\{\mathrm{e}^{-y\xi} + \frac{\ell^2\xi^2-\ell'\xi}{\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}-(1+\ell^2\xi^2)}\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\cos x\xi\,\mathrm{d}\xi$$

$$\mu_{xyz}(x,y) = \sigma_0\alpha\ell^2\int_0^\infty \xi J_1(\alpha\xi)\left\{\mathrm{e}^{-y\xi} + \frac{(\ell^2\xi^2-\ell'\xi)\sqrt{\xi^2+\frac{1}{\ell^2}}}{\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}-(1+\ell^2\xi^2)}\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\sin x\xi\,\mathrm{d}\xi$$

$$\mu_{yxz}(x,y) = -\sigma_0\alpha\int_0^\infty (\ell'-\ell^2\xi)J_1(\alpha\xi)\left\{\mathrm{e}^{-y\xi} + \frac{\ell^2\xi\sqrt{\xi^2+\frac{1}{\ell^2}}-\ell'\xi}{\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}-(1+\ell^2\xi^2)}\,\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\sin x\xi\,\mathrm{d}\xi$$

$$\mu_{yyz}(x,y) = -\sigma_0\alpha\int_0^\infty \xi^{-1}(\ell'\xi-\ell^2\xi^2)J_1(\alpha\xi)\left\{\mathrm{e}^{-y\xi}-\mathrm{e}^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\cos x\xi\,\mathrm{d}\xi$$

$$\varepsilon_{xz}(x,y) = -\frac{1}{2}\frac{\sigma_0\alpha}{G}\int_0^\infty \xi^{-1}J_1(\alpha\xi)\left\{e^{-y\xi} - \frac{\ell^2\xi^2-\ell'\xi}{1+\ell^2\xi^2-\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}}\,e^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\sin x\xi\,d\xi$$

$$\varepsilon_{yz}(x,y) = -\frac{1}{2}\frac{\sigma_0\alpha}{G}\int_0^\infty J_1(\alpha\xi)\left\{e^{-y\xi} - \frac{\ell\xi-\frac{\ell'}{\ell}}{\sqrt{1+\ell^2\xi^2}-\frac{\ell'}{\ell}}\,e^{-y\sqrt{\xi^2+\frac{1}{\ell^2}}}\right\}\cos x\xi\,d\xi \quad y \geqslant 0. \tag{34}$$

The above solutions indicates that the stresses are singular at the crack tip as in the classical LEFM theory, however, the strain field is finite everywhere in contrast to the classical theory which predicts infinite strain at the crack tip. Next, taking into account the third of the above equations, we apply the well-known Irwin's condition at the crack tip to obtain the stress intensity factor (SIF).

$$\begin{aligned} K_{\mathrm{III}} &= \lim_{x\to\alpha+}[\sqrt{2\pi(x-\alpha)}\sigma_{yz}(x,0)] \\ &= \lim_{x\to\alpha+}\sqrt{2\pi(x-\alpha)}(-\sigma_0\alpha)\int_0^\infty J_1(\alpha\xi)\cos x\xi\,d\xi = \sigma_0\sqrt{\pi\alpha} \end{aligned} \tag{35}$$

which agrees with the well-known LEFM result.

It can be shown that the faces of the crack do not completely close at its "physical tip", i.e. at $x = \alpha$, but they extend beyond it (strictly speaking at infinity) to close smoothly for sufficiently low ℓ'/ℓ-ratio as will be discussed in a next paragraph. As mentioned earlier in Altan and Aifantis (1992b) this feature of the above solution indicates that gradient elasticity substantiates Barenblatt's (1962) "cohesive zone" theory without explicitly postulating the elimination of the stress singularity at the crack tip.

Problem II: *clamped crack tips*

In this special half-plane $y \geqslant 0$ problem, the crack occupying the line segment $-\alpha < x < \alpha$, $y = 0$ will be subjected to the following mixed boundary conditions (derived from the virtual work principle)

$$\left.\begin{array}{ll} \sigma_{yz}(x,0) = -\sigma_0 & \\ \mu_{yyz}(x,0) = 0 & 0 \leqslant x < \alpha \\ u_z(x,0) = 0 & \\ \kappa_{yyz}(x,0) = 0 & \alpha < x < \infty \end{array}\right\} \tag{36}$$

and the same regularity conditions at infinity as in problem I of the previous paragraph.

Substitution of eqns (11) into (36), renders

$$\begin{array}{l} \left.\begin{array}{l} \dfrac{\partial u_z}{\partial y} - \ell^2\nabla^2\dfrac{\partial u_z}{\partial y} = -\sigma_0 \\ \ell'\dfrac{\partial u_z}{\partial y} + \ell^2\dfrac{\partial^2 u_z}{\partial y^2} = 0 \end{array}\right\} \quad 0 \leqslant x < \alpha \\ \left.\begin{array}{l} u_z = 0 \\ \dfrac{\partial^2 u_z}{\partial y^2} = 0 \end{array}\right\} \quad \alpha < x < \infty \end{array} \tag{37}$$

where here and henceforth $u_z(x,0)$ is denoted as u_z.

In fact, the last of conditions (37) is automatically satisfied considering that u_z is anti-symmetric with respect to y and u_z is zero outside the crack region. Furthermore, substituting (22) for u_z in the above equations we are led to the following system of dual integral equations for $B(\xi)$, since $A(\xi)$ was obtained previously as the solution of the first of equations (37).

$$\left.\begin{aligned} &\int_0^\infty \left[(1+\ell^2\xi^2)-\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}\right]B(\xi)\cos x\xi\,\mathrm{d}\xi = \sqrt{\frac{\pi}{2}}\frac{\sigma_0\alpha}{G} \\ &\qquad\times\int_0^\infty \xi^{-1}(\ell'\xi-\ell^2\xi^2)J_1(\alpha\xi)\cos x\xi\,\mathrm{d}\xi \qquad 0\leqslant x<\alpha \\ &\int_0^\infty B(\xi)\cos x\xi\,\mathrm{d}\xi = -\sqrt{\frac{\pi}{2}}\frac{\sigma_0\alpha}{G}\int_0^\infty \xi^{-1}J_1(\alpha\xi)\cos x\xi\,\mathrm{d}\xi \qquad \alpha<x<\infty \end{aligned}\right\}. \tag{38}$$

The objective now is to transform the dual integral equations (38) into a regular integral equation of a standard form. This is accomplished by taking the following Riemann–Liouville fractional integral representation (Erdelyi, 1954) for $u_z^g(x,0)$

$$u_z^g(x,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty B(\xi)\cos x\xi\,\mathrm{d}\xi = \int_x^\alpha \varphi(t)\sqrt{t^2-x^2}\,\mathrm{d}t \quad 0\leqslant x<\alpha \tag{39}$$

where φ is a sectionally continuous function which may exhibit weak singularities in $[0,\alpha]$ and it is allowed to depend on ℓ, ℓ' and α.

Applying the Fourier cosine inversion formula to the above equation, the following result is derived

$$B(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\alpha \cos x\xi\,\mathrm{d}x\int_x^\alpha \varphi(t)\sqrt{t^2-x^2}\,\mathrm{d}t \quad 0<\xi<\infty. \tag{40}$$

By using the extended Dirichlet's formula for reversing the order of integration of functions exhibiting weak singularities inside the closed domain of integration (Whittaker and Watson, 1948) and by using the well-known identity (Gradshteyn and Ryzhik, 1980)

$$\int_0^t (t^2-x^2)^{1/2}\cos x\xi\,\mathrm{d}x = \frac{\pi}{2}\frac{t}{\xi}J_1(t\xi) \quad 0<\xi<\infty \tag{41}$$

the following result is obtained

$$B(\xi) = \sqrt{\frac{\pi}{2}}\xi^{-1}\int_0^\alpha t\varphi(t)J_1(t\xi)\,\mathrm{d}t \quad 0<\xi<\infty. \tag{42}$$

In summary, $u_z^g(x,0)$ must vanish for $x>\alpha$ and satisfy eqn (40) for $0\leqslant x<\alpha$. The function $B(\xi)$ is then given by eqn (42). Finally, on noting that

$$\left.\begin{aligned} &\int_0^\infty \xi^{-1} J_1(\alpha\xi)\cos x\xi\,\mathrm{d}\xi = 0 && x > \alpha \\ &\int_0^\alpha t\varphi(t)\,\mathrm{d}t \int_0^\infty \xi^{-1} J_1(t\xi)\cos x\xi\,\mathrm{d}\xi = 0 && x \geqslant t \end{aligned}\right\} \tag{43}$$

it is concluded that the condition of zero displacement ahead of the crack tip [second of relations (38)] is satisfied identically.

Next, the expression for $B(\xi)$ given in eqn (42) is introduced in the first of equations (38), x is replaced by x' and an integration with respect to x' over the range $0 \leqslant x' \leqslant x$ $(0 \leqslant x < \alpha)$ is performed to yield

$$\int_0^\alpha t\varphi(t)\,\mathrm{d}t \int_0^\infty \left[\frac{1+\ell^2\xi^2 - \dfrac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}}{\ell^2\xi^2}\right] J_1(t\xi)\sin x\xi\,\mathrm{d}\xi$$

$$= \frac{\sigma_0\alpha}{G\ell^2}\int_0^\infty \left[\frac{\ell'-\ell^2\xi}{\xi}\right] J_1(\alpha\xi)\sin x\xi\,\mathrm{d}\xi \quad 0 \leqslant x < \alpha. \tag{44}$$

The above equation is a linear Fredholm integral equation of the first kind

$$\int_0^\alpha K_0(x,t)t\varphi(t)\,\mathrm{d}t = F(x) \quad 0 \leqslant x < \alpha \tag{45}$$

where

$$K_0(x,t) = \int_0^\infty \left(\frac{1+\ell^2\xi^2 - \dfrac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}}{\ell^2\xi^2}\right) J_1(t\xi)\sin x\xi\,\mathrm{d}\xi \tag{46}$$

is the kernel, and

$$F(x) = \frac{\sigma_0\alpha}{G\ell^2}\int_0^\infty \left(\frac{\ell'-\ell^2\xi}{\xi}\right) J_1(\alpha\xi)\sin x\xi\,\mathrm{d}\xi \quad 0 \leqslant x < \alpha \tag{47}$$

is the free term. Fredholm integral equations of the first kind are, in general, difficult to solve due to their ill-posedness from a conventional standpoint, and furthermore it is not guaranteed that they have a unique solution (Tricomi, 1957). However, it is possible to reduce the solution of (45) to that of a more tractable Fredholm integral equation of the second kind with a simpler kernel.

First, observe that

$$K_0(t,x) = K_1(t,x) + K_2(t,x) \tag{48}$$

where

$$K_1(t,x) = \int_0^\infty J_1(t\xi)\sin x\xi\,\mathrm{d}\xi \tag{49}$$

and

$$K_2(t,x) = \int_0^\infty \beta(\ell\xi) J_1(t\xi) \sin x\xi \, d\xi \tag{50}$$

with

$$\beta(\xi) = \frac{1 - \frac{\ell'}{\ell}\sqrt{1+\xi^2}}{\xi^2} \quad 0 < \xi < \infty. \tag{51}$$

Taking into consideration the value of the discontinuous integral

$$K_1(t,x) = \frac{x}{t} \frac{1}{\sqrt{t^2 - x^2}} \quad t > x, \tag{52}$$

eqn (44) becomes

$$x \int_x^\alpha \frac{\varphi(t)}{\sqrt{t^2 - x^2}} dt + \int_0^\alpha t\varphi(t) \, dt \int_0^\infty \beta(\ell\xi) J_1(t\xi) \sin x\xi \, d\xi = F(x) \quad 0 \leqslant x < \alpha \tag{53}$$

which has the form of the familiar Abel integral equation

$$\int_x^\alpha \frac{\varphi(t)}{\sqrt{t^2 - x^2}} dt = f(x) \quad 0 \leqslant x < \alpha \tag{54}$$

where

$$f(x) = \left\{ \frac{F(x)}{x} - \int_0^\alpha \tau\varphi(\tau) \, d\tau \int_0^\infty \xi\gamma(x\xi)\beta(\ell\xi) J_1(\tau\xi) \, d\xi \right\} \quad 0 \leqslant x < \alpha \tag{55}$$

and

$$\gamma(\xi) = \frac{\sin\xi}{\xi}. \tag{56}$$

The integral equation (54) has the solution (Sneddon, 1966)

$$\varphi(t) = -\frac{2}{\pi} \frac{d}{dt} \int_t^\alpha \frac{x f(x) \, dx}{\sqrt{x^2 - t^2}} \quad 0 \leqslant t \leqslant \alpha. \tag{57}$$

Since $f(x)$ is differentiable, integration by parts and subsequent differentiation, yields

$$\varphi(t) = -\frac{2t}{\pi} \int_t^\alpha \frac{1}{\sqrt{x^2 - t^2}} \frac{d}{dx}[f(x)] \, dx \quad 0 \leqslant t \leqslant \alpha. \tag{58}$$

Taking into account the well-known identities (Gradshteyn and Ryzhik, 1980)

$$\int_t^\infty \frac{\sin yx\,\mathrm{d}x}{\sqrt{x^2-t^2}} = \frac{\pi}{2}J_0(yt), \quad \int_0^\xi yJ_0(ty)\,\mathrm{d}y = \frac{\xi}{t}J_1(t\xi),$$

$$\text{and} \quad \int y\sin y\,\mathrm{d}y = \sin y - y\cos y \tag{59}$$

it is found that

$$\int_t^\alpha \frac{1}{\sqrt{x^2-t^2}}\frac{\mathrm{d}}{\mathrm{d}x}[\gamma(x\xi)]\,\mathrm{d}x = -\frac{\pi}{2t}J_1(t\xi). \tag{60}$$

Moreover, the value of the Riemann–Liouville operation (58) on the function $F(x)$ can be found by virtue of the relation

$$\int_0^\infty \xi J_1(\alpha\xi)J_1(t\xi)\,\mathrm{d}\xi = \frac{\delta(\alpha-t)}{(\alpha t)^{1/2}} \quad 0 \leqslant t \leqslant \alpha \tag{61}$$

where $\delta(x)$ is the usual Dirac delta function, and by also utilizing the value given by Gubler for the infinite Weber–Schafheitlin integral (Watson, 1958)

$$I(t,\alpha) = \int_0^\infty J_1(\alpha\xi)J_1(t\xi)\,\mathrm{d}\xi = \frac{t}{2\alpha^2}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{t^2}{\alpha^2}\right) \tag{62}$$

with ${}_2F_1(a,b;c;z)$ to be the hypergeometric function. Since in our case $c-a-b=0$, the series representation of the hypergeometric function

$${}_2F_1(a,b;c;z) = 1+\frac{a\cdot b}{c\cdot 1}z+\frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2+\cdots \tag{63}$$

converges absolutely throughout $[0,1)$, while an examination of (62) reveals that in the limit as $z \to 1$ we have

$$I(t,\alpha) = -\frac{1}{\pi}\frac{1}{t}\left[\log\left(1-\frac{t^2}{\alpha^2}\right)+O(1)\right] \quad 0 < t < \alpha. \tag{64}$$

Equation (58) is equivalent to an inhomogeneous linear Fredholm integral equation of the second kind with a weakly singular kernel which belongs to the Lebesgue class L^2 in the square $[0,\alpha]\times[0,\alpha]$ and a free term which belongs to L^1-class in $[0,\alpha]$ (Appendix II),

$$\varphi(t) = -\ell^{-2}\int_0^\alpha \tau\varphi(\tau)\,\mathrm{d}\tau\int_0^\infty \xi^{-1}\left[1-\frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}\right]J_1(t\xi)J_1(\tau\xi)\,\mathrm{d}\xi$$
$$+\frac{1}{2}\frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{t}{\alpha}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{t^2}{\alpha^2}\right)-\frac{\sigma_0}{G}\frac{\alpha^{1/2}}{t^{1/2}}\delta(\alpha-t) \quad 0 \leqslant t \leqslant \alpha. \tag{65}$$

In general, both existence and uniqueness of solution of eqn (65) are assured by Fredholm theory provided that ℓ^{-2} is not an eigenvalue of the homogeneous equation associated with (65) [Tricomi 1957]. Solution of the above Fredholm integral equation by employing Liouville–Neumann method of successive approximations (Whittaker and Watson, 1948) will not give, in general, a complete picture due to convergence problems (Appendix B). However, according to Fredholm's first theorem, the above inhomogeneous equation has the following unique solution which is unbounded at $t=\alpha$ but belongs to L^1-class in $[0,\alpha]$,

$$\varphi(t) = g(t) + \lambda \int_0^\alpha \Gamma(t, \tau; \lambda) g(\tau)\, \mathrm{d}\tau \quad 0 \leqslant t \leqslant \alpha \tag{66}$$

where

$$\lambda = -\frac{1}{\ell^2}, \quad g(t) = \frac{1}{2}\frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{t}{\alpha}\,{}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 2; \frac{t^2}{\alpha^2}\right) - \frac{\sigma_0}{G}\frac{\alpha^{1/2}}{t^{1/2}}\delta(\alpha - t) \quad 0 \leqslant t \leqslant \alpha \tag{67}$$

and $\Gamma(t, \tau; \lambda)$ is the resolvent kernel of the Fredholm integral equation (65) given as the quotient of two power series in λ (Tricomi, 1957)

$$\Gamma(t, \tau; \lambda) = \frac{\sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} C_p(t, \tau)}{\sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} c_p} \tag{68}$$

with

$$c_0 = 1, \quad c_p = \int_0^\alpha C_{p-1}(s, s)\, \mathrm{d}s, \quad C_p(t, \tau) = c_p K(t, \tau) - p \int_0^\alpha K(t, x) C_{p-1}(x, \tau)\, \mathrm{d}x,$$

$$K(t, \tau) = \tau \int_0^\infty \xi^{-1}\left[1 - \frac{\ell'}{\ell}\sqrt{1 + \ell^2\xi^2}\right] J_1(t\xi) J_1(\tau\xi)\, \mathrm{d}\xi. \tag{69}$$

Both series in (68) converge for all values of λ and according to the generalized Liouville theorem the resolvent exists for all λ. By means of the shifting (or sampling) property of the delta function

$$\int \delta(x - x_0)\varphi(x_0)\, \mathrm{d}x = \varphi(x_0)$$

and relationship (67), relationship (66) takes the following form

$$\varphi(t) = \frac{\sigma_0}{G}\left[-\left(\frac{\alpha}{t}\right)^{1/2}\delta(\alpha - t) + \ell^{-2}\Gamma(t, \alpha; -\ell^{-2}) + \frac{1}{2}\frac{\ell'}{\ell^2}\frac{t}{\alpha}\,{}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 2; \frac{t^2}{\alpha^2}\right)\right.$$
$$\left. - \frac{1}{2\alpha}\frac{\ell'}{\ell^4}\int_0^\alpha \tau\,{}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 2; \frac{t^2}{\alpha^2}\right)\Gamma(t, \tau; -\ell^{-2})\, \mathrm{d}\tau\right] \quad 0 \leqslant t \leqslant \alpha. \tag{70}$$

By using again the shifting property of the delta function, and recalling equations (17) and (40), it can be easily found that the leading term of the above representation of the function φ cancels out the displacements predicted by the classical theory. Thus, the shape of the crack is no longer elliptical as predicted by LEFM, but given by the formula

$$u_z(x, 0) = \int_x^\alpha \varphi^*(t)\sqrt{t^2 - x^2}\, \mathrm{d}t \tag{71}$$

where we have set

$$\varphi^*(t) = \frac{\sigma_0}{G}\left[\ell^{-2}\Gamma(t,\alpha;-\ell^{-2})+\frac{1}{2}\frac{\ell'}{\ell^2}\frac{t}{\alpha}{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{t^2}{\alpha^2}\right)\right.$$
$$\left.-\frac{1}{2\alpha}\frac{\ell'}{\ell^4}\int_0^\alpha \tau_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{t^2}{\alpha^2}\right)\Gamma(t,\tau;-\ell^{-2})\,\mathrm{d}\tau\right] \quad (72)$$

with $\varphi^*(t)$ being a L^2-function in $[0,\alpha]$.

After integration by parts of expression (71) and by an asymptotic analysis of the solution close to the crack tip, we obtain (Appendix C)

$$u_z(r,0)|_{r\to 0} = \frac{2\sqrt{2}}{3}\alpha^{1/2}\phi^*(\alpha-\eta)r^{3/2}+O(r^{5/2})+\varepsilon(\eta) \quad (73)$$

where $r = \alpha-x$. In Eqn (73) η is a small length with respect to the semi-crack length α, in order to remove the weak logarithmic singularity of φ^* at $t = \alpha$. The following upper bound has been found which gives the desired accuracy as a function of η (Appendix C)

$$|\varepsilon(\eta)| = \left|\int_{\alpha-\eta}^{\alpha}\varphi^*(t)\sqrt{t^2-x^2}\,\mathrm{d}t\right| \leqslant N(\eta^*)\,; \quad \eta^* = \frac{1}{\pi}\frac{\eta}{\alpha}$$

with

$$N(\eta^*) = \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\alpha^2|-0.15\eta^*+\eta^*\log\eta^*| \to 0 \quad \text{as } \eta^* \to 0. \quad (74)$$

For example, if we take $\eta^* = 1.6E-3$, the accuracy is $0.0105(\sigma_0\ell'\alpha^2)/(G\ell^2)$. According to eqn (73), the crack shape predicted by gradient elasticity with surface energy is described by the equation

$$\left(\frac{x}{\alpha}\right)^2+\left(\frac{u_z}{\beta}\right)^{2/3} = 1 \quad (75)$$

where $\beta = u_z(0,0)$. That is, the crack lips form a cusp of the first kind with zero enclosed angle and zero first derivative of the displacement at the crack tip (Fig. 2). It is noted that LEFM predicts infinite strain at the crack tip. This fact indicates that gradient elasticity theory with or without the surface energy term predicts the same crack shape with Barenblatt's (1962) "cohesive-zone" theory without requiring an extra assumption on the existence of interatomic forces at the outset beyond those implied by the gradient terms in the generalized constitutive equation.

The strain energy release rate (or, alternatively the crack driving force) of the mode-III crack can be computed if on the extension of the crack we introduce an imaginary cut whose faces are acted upon by the stresses produced in the intact medium near the crack tip by the load exerted on the original crack surfaces. The required energy release rate G_{III} is computed by considering the virtual work done by stress σ_{yz} and the double stress μ_{yyz} on the displacement u_z, and the displacement gradient $\partial u_z/\partial y$, respectively, for a virtual infinitesimal advancement of the crack tip by a distance δ, i.e.

$$G_{\mathrm{III}} = 2\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta \frac{1}{2}\left[\sigma_{yz}(\alpha+\delta-\beta,0)u_z(\alpha-\beta,0)+\mu_{yyz}(\alpha+\beta,0)\frac{\partial u_z}{\partial y}(\alpha-\beta,0)\right]\mathrm{d}\beta. \quad (76)$$

In polar coordinates (r,θ) from the crack tip, and by virtue of eqns (35) and (73), the quantities appearing in the above integrand have the following asymptotic expansions

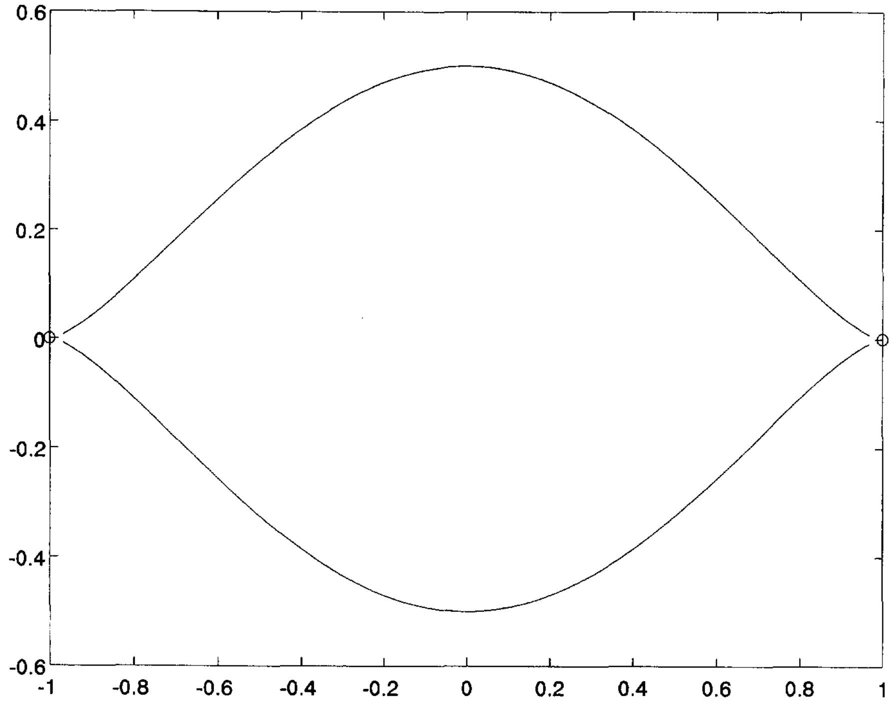

Fig. 2. Crack with tips in the form of cusps of first kind ($\alpha = 1$, $\beta = 1/2$).

$$
\begin{aligned}
\sigma_{yz}(r,\theta) &= \frac{1}{\sqrt{\pi}}\frac{K_{\mathrm{III}}}{\sqrt{r}}\cos\frac{\theta}{2} + o(r^{1/2}) \\
u_z(r,\theta) &= -\frac{k_3 r^{3/2}}{G}\sin\frac{3\theta}{2} + o(r^{5/2}) \\
\mu_{yyz}(r,\theta) &= -\frac{3\ell^2}{4}\frac{k_3}{\sqrt{r}}\sin\frac{\theta}{2} - \frac{3\ell'}{2}k_3\sqrt{r} \\
&\quad \times\cos\frac{\theta}{2} + o(r^{3/2}) \\
\frac{\partial u_z}{\partial y}(r,\theta) &= -\frac{3}{2}\frac{k_3}{G}\sqrt{r}\cos\frac{\theta}{2} + o(r^{3/2})
\end{aligned} \tag{77}
$$

where

$$
k_3 = \frac{2\sqrt{2}G}{3}\alpha^{1/2}\varphi^*(\alpha-\eta). \tag{78}
$$

Upon using eqn (77), the integration of eqn (76) can be carried out explicitly to yield

$$
\frac{G_{\mathrm{III}}}{G_{\mathrm{III}}^c} = \frac{3\sqrt{\pi}}{8}\frac{K_{\mathrm{III}}k_3}{G}\lim_{\delta\to 0}\delta = 0 \tag{79}
$$

where $G_{\mathrm{III}}^c = (\pi/2)(\sigma_0^2\alpha/G)$ is the LEFM solution. Thus, in gradient elasticity with surface energy theory there is no contribution to the work rate from the "holding force" on the crack extension, as is also predicted by Barenblatt's (1962) "cohesive zone" theory. In Griffith's theory, on the other hand, the work rate comes entirely from the "holding force" on the extension, and this is possible only because there is a suitable stress singularity and displacement square root radius dependence, and the limitations of the linear theory to small strain are not considered. Alternatively, the energy release rate in gradient elasticity with surface energy can be found from the relation which gives the strain energy W required to form a pressurized crack, i.e.

$$W = \sigma_0 \int_0^\alpha u_z(x,0)\,\mathrm{d}x = \sigma_0 \int_0^\alpha \mathrm{d}x \int_x^\alpha \varphi^*(t)\sqrt{t^2-x^2}\,\mathrm{d}t = \sigma_0 \int_0^\alpha \varphi^*(t)\,\mathrm{d}t \int_0^t \sqrt{t^2-x^2}\,\mathrm{d}x,$$

or

$$W = \frac{\pi}{4}\frac{\sigma_0^2}{G}\int_0^\alpha t^2\hat{\varphi}(t)\,\mathrm{d}t \tag{80}$$

with $\hat{\varphi}(t) = \varphi^*(t)/(\sigma_0/G)$. The energy release rate is then given by

$$G_{\mathrm{III}} = \frac{\partial W}{\partial \alpha} = \sigma_0 \frac{\partial}{\partial \alpha}\left[\int_0^\alpha u_z(x,0)\,\mathrm{d}x\right] = \frac{\pi}{4}\frac{\sigma_0^2}{G}\frac{\partial}{\partial \alpha}\int_0^\alpha t^2\hat{\varphi}(t)\,\mathrm{d}t. \tag{81}$$

Finally, the asymptotic expansions of the strain components in the crack tip region give

$$\left.\begin{aligned} \varepsilon_{xz}(r,\theta) &= -\frac{3}{4}\frac{k_3}{G}\sqrt{r}\sin\frac{\theta}{2} + o(r^{3/2}) \\ \varepsilon_{yz}(r,\theta) &= -\frac{3}{4}\frac{k_3}{G}\sqrt{r}\cos\frac{\theta}{2} + o(r^{3/2}) \end{aligned}\right\}. \tag{82}$$

The above results demonstrate that the strain field remains finite at the crack tip region in contrast to LEFM and therefore they may allow for the development of more realistic crack initiation theories based on "maximum strain" criteria.

4. NUMERICAL RESULTS AND DISCUSSION

First, we introduce the dimensionless variables m and m' defined as

$$m = \ell/\alpha, \quad m' = \ell'/\alpha \tag{83}$$

and the normalized crack displacement

$$u_z^\circ = \frac{u_z}{(\sigma_0/G)}. \tag{84}$$

(a) Problem I: *unclamped crack tips*

Since the integrals appearing in expressions (34) for the displacement and strain tensor components cannot be evaluated explicitly we calculate them numerically. In order to deal with the strongly oscillatory character of the integrands especially for large values of x, Filon's method was employed (Davis and Rabinowitz, 1984). Furthermore, it was found that all integral expressions display good convergence characteristics, and, thus, the range of integration becomes finite.

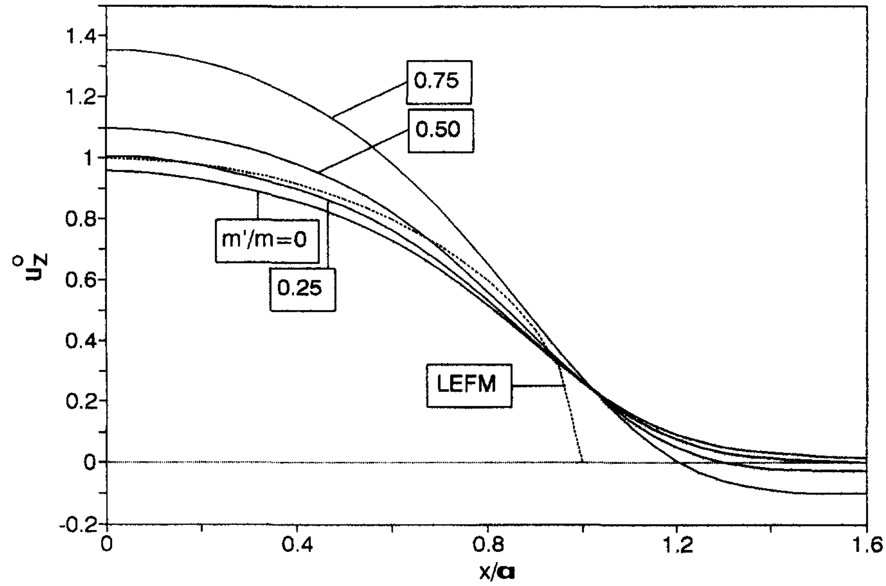

Fig. 3. LEFM elliptic crack shape ($m = 0$). Effect of the material length ratio m'/m on the normalized displacement along the pressurized crack with unclamped tips for $m = 0.2$.

In Fig. 3 the distributions of the normalized crack displacement, referring to the upper right part of a crack, for $m = 0.2$ and for various values of the relative surface energy parameter m', are shown. The crack displacement distribution predicted by LEFM is also superimposed in the same figure. It is clear that for $m > 0$ the cracks do not completely close at the tip ($x/\alpha = 1$) of the physical region, that is the region where the traction is specified $[0, \alpha]$. As is also apparent from Fig. 3 the crack displacements in the physical crack region increase monotomically from the value of $m' = 0$ on, i.e. as the ratio m'/m increases. It is worth noting that for low m'/m-values the cracks close smoothly to form a cusp, whereas for sufficiently large m'/m-values the crack surfaces exhibit negative anti-plane shear displacement at the process zone suggesting that, at least for these values of m', the specific boundary value problem is physically not meaningful.

(b) Problem II: *clamped crack tips*

As far as the second mixed boundary value problem is concerned, the multiple integrals appearing in the expressions (69) were evaluated by a Gauss–Legendre numerical quadrature scheme. In order to demonstrate the nice convergence behavior of the normalized displacement at the mid-point of the crack $u_z^o(0, 0)$ given by the relationships (71) and (84) for various values of p, which refers to the number of terms in the truncated series (68), and of the relative volume energy parameter m, Table 1 is constructed. Accordingly, Fig. 4 shows the convergence of the solution for the semi-crack profile in the upper half-plane for

Table 1. Effect of the relative volume energy parameter m on the convergency characteristics of the solution of the Fredholm integral equation for $m' = 0$

	m						
p	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0	16.666670	4.166667	1.851852	1.041667	0.666667	0.462963	0.340136
1	5.376926	1.636208	0.892593	0.594207	0.431840	0.329296	0.259184
2	2.690611	1.023052	0.677680	0.508172	0.394461	0.311874	0.250528
3	1.663240	0.828621	0.630101	0.495324	0.390596	0.310577	0.250046
4	1.198415	0.774852	0.623048	0.494213	0.390381	0.310527	0.250033
5	0.989153	0.764663	0.622437	0.494159			
6	0.925941	0.763574	0.622408				
7	0.924511	0.763268					

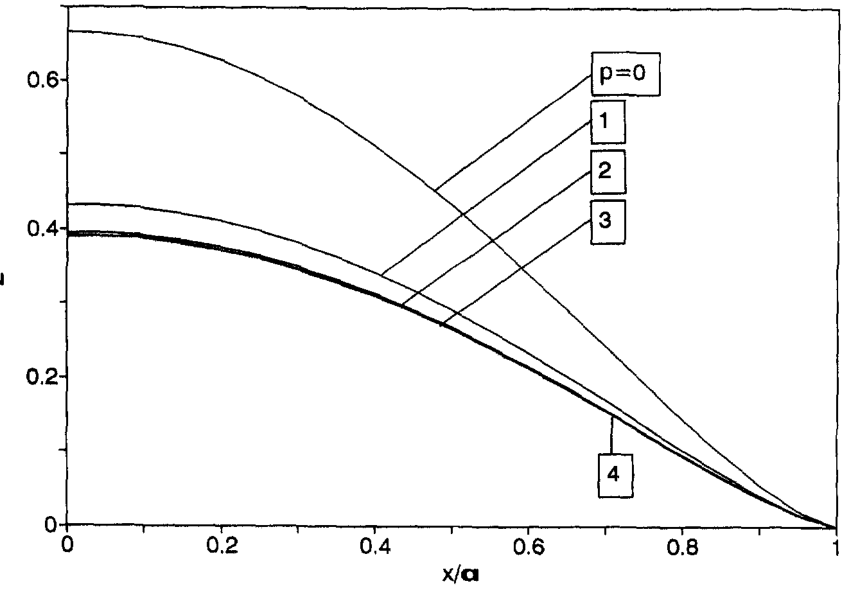

Fig. 4. Convergency of the solution giving the normalized displacement along the crack with clamped tips for various values of p ($m = 0.5$, $m' = 0$).

$m = 0.5$ and $m'/m = 0$. It can be seen that the faces of the crack tend smoothly to the x-axis to form a cusp of zero contained angle, as it was also shown above by the asymptotic analysis. In Fig. 5 the results which refer to the dependence of the normalized displacement at the mid-point of the crack upon the relative volume energy parameter m together with those predicted by Ru and Aifantis (1993), are presented. From this figure it is noted that the solution of the Fredholm equation definitely tends to $u_z(0,0)/u_z^c(0,0) = 1$ for $m \to 0$, while the respective solution of the ordinary differential equation (o.d.e.) given by Ru and Aifantis (1993) tends to the classical crack displacement solution for $m = 0.23$.

In Fig. 6 it is shown that the semi-crack profile predicted by the Fredholm integral equation approaches the LEFM solution in a global sense as $m \to 0$; however, it is characterized always by a cusp at the tip region. In contrast, the solution of the o.d.e. always gives a finite and non-zero value of the displacement gradient at the crack tip (Ru and

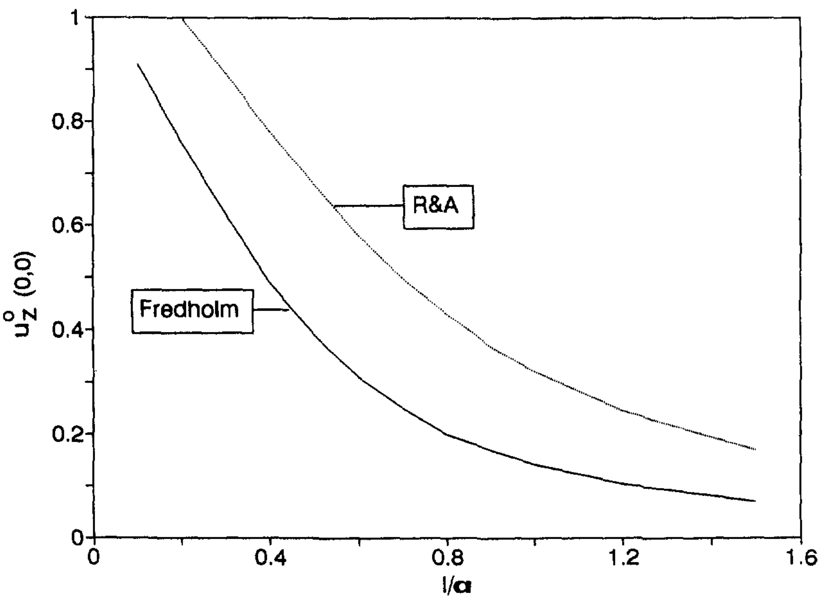

Fig. 5. Comparison of the present solution with that given by Ru and Aifantis (1993) for the clamped crack with $m' = 0$.

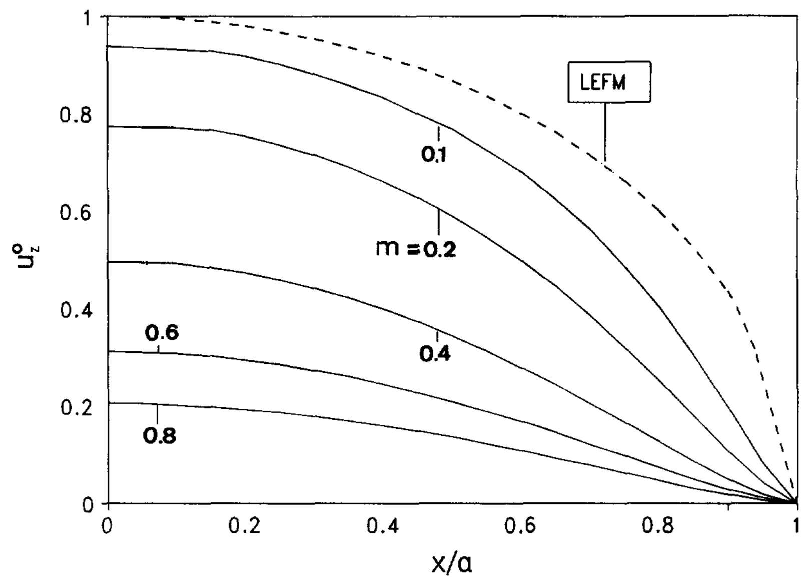

Fig. 6. LEFM elliptic crack shape ($m = 0$). Present theory predictions concerning the effect of the relative volume energy term on the displacement of the cusping crack with clamped tips ($m' = 0$).

Aifantis, 1993). As is apparent from Figs 5 and 6, the crack displacement diminishes monotonically, or alternatively the crack becomes "stiffer", in comparison to the classical theory, as m increases.

Furthermore, keeping the value of the relative volume energy parameter m constant, the crack stiffening effect becomes more pronounced as the relative surface energy parameter m' increases in the domain $[0, m)$ as is evident from Fig. 7. We should remark that from energy considerations the parameter m' can assume negative values (Appendix A). The effect of a negative m' is shown here in Fig. 7 leading to a more "compliant" crack. This in general is a welcome property of the mathematical model as far as a description of experimental data is concerned.

The crack stiffening effect due to the increase of the dimensionless value m'/m for various values of the relative volume energy parameter m is displayed in Fig. 8. As it is also shown in the same figure, $u_z^\circ \to 0$ as $m'/m \to 1$.

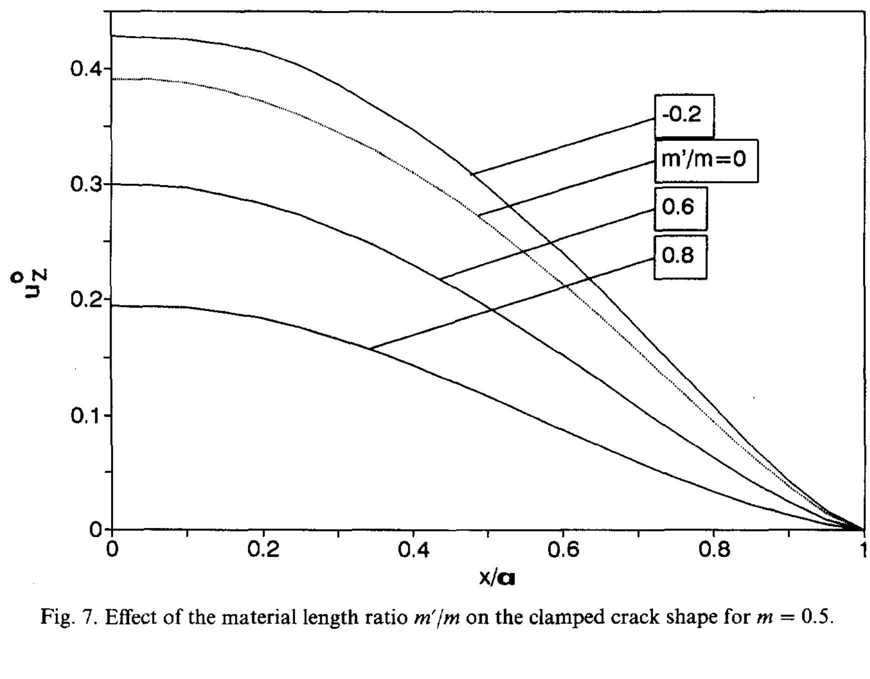

Fig. 7. Effect of the material length ratio m'/m on the clamped crack shape for $m = 0.5$.

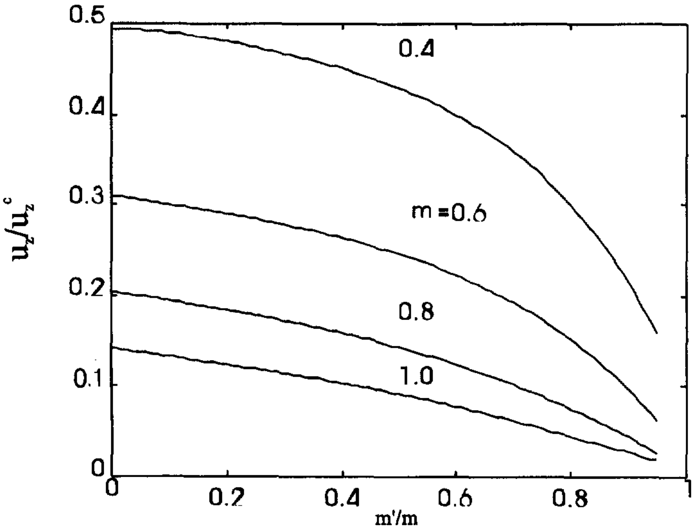

Fig. 8. Effect of the material length ratio m'/m on the mid-point normalized displacement of the clamped-tips crack for various values of m.

As expected, this crack stiffening due to the characteristic material lengths ℓ and $\ell'(>0)$ of the structured medium is responsible for lower energy release rates during crack propagation. This is clearly demonstrated in Fig. 9 where it can be seen that as m'/m increases for a given volume energy parameter, the crack driving force decreases appreciably.

In view of Griffith's rupture criterion $G \geqslant 2\gamma$, where $2\gamma[FL^{-1}]$ is a material property, the so-called "specific fracture energy", the above results suggest that a higher external load as compared to that of the classical case must be exerted on the crack or on the remote boundaries, in order to propagate it in a material with microstructure. This prediction is consistent with the experimental result that the specific fracture energy 2γ of rock or rock-like materials, e.g. marble and ceramics, is much greater than the weighted average of the fracture energies of their constituent minerals (Friedman *et al.*, 1972; Atkinson, 1984; and others). For example, in glass and minerals $2\gamma \approx 1$–$10\ \mathrm{J/m^2}$, while in rocks 2γ can be two–three orders of magnitude greater. According to the present theory this apparent increase of fracture energy is due to the appreciable decrease of G_{III} in a structured medium with $G_{\mathrm{III}} \to 0$ as $m'/m \to 1$.

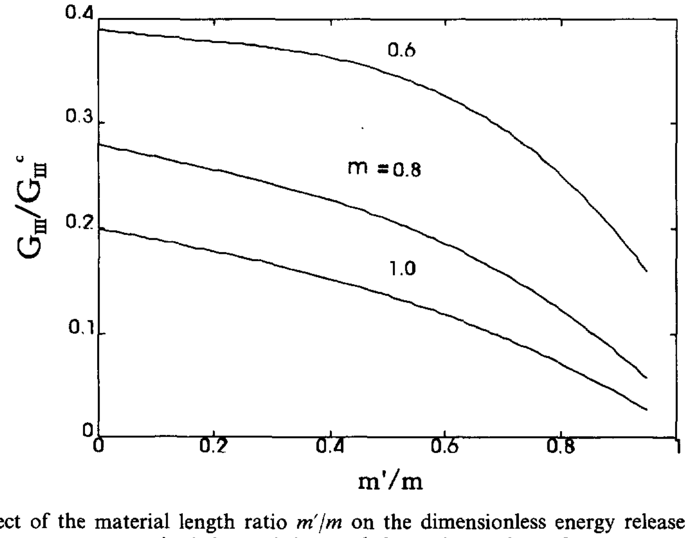

Fig. 9. Effect of the material length ratio m'/m on the dimensionless energy release rate of the pressurized clamped-tips crack for various values of m.

From the above it is clear that the determination of the two characteristic material lengths, ℓ, ℓ', constitutes a formidable experimental challenge, at least for the specific case examined here, that is the anti-plane shear case, through precise monitoring of the crack profile in torsion experiments of pre-notched cylindrical bars of rocks with known granulometry.

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APPENDIX A

For demonstration purposes we consider first the 1D example of a tension bar of length L in a clamped end-free end configuration with the load acting on its free end along the x-axis. In the uniaxial case the strain energy of the tension bar is defined as follows

$$W = \frac{1}{2}\int_0^L E(\varepsilon^2 + \ell^2 \nabla\varepsilon\nabla\varepsilon)\,\mathrm{d}x + \frac{1}{2}[E\ell'\varepsilon^2]_0^L \tag{A.1}$$

where

$$\nabla\varepsilon = \frac{d\varepsilon}{dx} \tag{A.2}$$

denotes the strain gradient. Casal's elastic strain energy ansatz (A.1) consists of two terms: (a) a "volume energy" term which includes the contribution of the strain gradient, and (b) a "surface energy" term. Accordingly, ℓ and ℓ' are material lengths related to volume and surface elastic strain energy, respectively.

Casal's expression (A.1) for the global elastic strain energy of the tension bar is recovered by introducing an appropriate anisotropic, linear elastic, restricted Mindlin continuum (Mindlin, 1964). Since in a restricted continuum the relative deformation vanishes, the variation of the strain energy density becomes

$$\delta w = \tau\delta\varepsilon + \mu\delta\nabla\varepsilon. \tag{A.3}$$

In this expression τ (Cauchy term) works on the strain, whereas μ (double stress) works on the strain gradient (Mindlin, 1964). In connection to the variation (A.3), Casal's model is equivalent with the following constitutive assumptions for the Cauchy- and double-stress

$$\tau = \frac{\partial w}{\partial \varepsilon} = E(\varepsilon + \ell'\nabla\varepsilon) \tag{A.4}$$

$$\mu = \frac{\partial w}{\partial \nabla\varepsilon} = E(\ell'\varepsilon + \ell^2\nabla\varepsilon). \tag{A.5}$$

From eqns (A.3)–(A.5) the elastic strain density is derived, resulting in the following expression

$$w = \tfrac{1}{2}E\{\varepsilon^2 + \ell^2\nabla\varepsilon\nabla\varepsilon\} + \tfrac{1}{2}E\ell'\nabla\varepsilon^2. \tag{A.6}$$

It turns out that for positive strain energy density ($w > 0$) in the considered 1D case, the material lengths are restricted, such that

$$-1 < \frac{\ell'}{\ell} < 1. \tag{A.7}$$

This means that if surface energy terms are included, then volume strain-gradient terms must also be included. It is worth noting that in Griffith's (1921) original theory of cracks only surface energy is considered, which is of course inadmissible in the sense of inequality (A.7), however, it is unambiguous when applied to the systems for which its basic premises are met. Altan and Aifantis (1992b) in analysing mode-III cracks have shown that in the tip region strain gradient terms are important. However, they have only considered the volumetric strain-gradient energy term ignoring the surface energy term, which is important within the present framework of elasticity theory with microstructure.

In the uniaxial case the following equilibrium conditions holds true for the total stress σ

$$\frac{d\sigma}{dx} = 0 \tag{A.8}$$

where

$$\sigma = \tau + \alpha \tag{A.9}$$

and α is the workless relative stress, which is equilibrated by the double stress

$$\alpha + \frac{d\mu}{dx} = 0. \tag{A.10}$$

The appropriate stress–strain relations of gradient-elasticity are finally derived from eqns (A.5), (A.6) and (A.9), (A.10) as follows

$$\sigma = E(\varepsilon - \ell^2\nabla^2\varepsilon). \tag{A.11}$$

The tension bar problem is thus defined by: (a) the constitutive eqns (A.5) and (A.11) for the double stress and the total stress, respectively; (b) the equilibrium condition (A.8) for the total stress, and (c) appropriate boundary conditions, which may be selected as follows

$$\begin{aligned} &\text{for } x = 0: \quad u = 0 \quad \text{and} \quad du/dx = 0 \\ &\text{for } x = L: \quad \sigma = \bar{\sigma} \quad \text{and} \quad \mu = 0 \end{aligned} \tag{A.12}$$

with u being the displacement of the bar.

The solution of the above problem gives an inhomogeneous strain distribution under constant stress. It reads

$$\varepsilon = \frac{\bar{\sigma}}{E}\left\{1 + \frac{1}{\left[1-\left(\frac{\ell}{\ell'}\right)^2\right]\mathrm{sh}\left(\frac{L}{\ell}\right)}\left[\mathrm{ch}\left(\frac{x}{\ell}\right) - \frac{\ell}{\ell'}\mathrm{sh}\left(\frac{x}{\ell}\right) - \frac{\ell}{\ell'}\mathrm{ch}\left(\frac{L-x}{\ell}\right) - \mathrm{sh}\left(\frac{L-x}{\ell}\right)\right]\right\}. \tag{A.13}$$

APPENDIX B

Consider the governing Fredholm integral equation of the second kind (65)

$$\varphi(t) + \frac{1}{\ell^2}\int_0^\alpha K(t,\tau)\varphi(\tau)\,\mathrm{d}\tau = g(t) \tag{B.1}$$

with

$$K(t,\tau) = \tau\int_0^\infty \xi^{-1}\left[1 - \frac{\ell'}{\ell}\sqrt{1+\ell^2\xi^2}\right]J_1(t\xi)J_1(\tau\xi)\,\mathrm{d}\xi \tag{B.2}$$

and $g(t)$ defined through eqn (67).

By introducing the variables

$$t^* = \frac{t}{\alpha}, \quad \tau^* = \frac{\tau}{\alpha}, \quad \xi^* = \xi\ell, \tag{B.3}$$

eqn (A.4) becomes

$$\varphi(\alpha t^*) + \ell^{-2}\int_0^1 K(t^*,\tau^*)\varphi(\alpha\tau^*)\alpha\,\mathrm{d}\tau^* = g(\alpha t^*) \tag{B.4}$$

with

$$K(t^*,\tau^*) = \alpha\tau^*\int_0^\infty \xi^{*-1}\left[1 - \frac{\ell'}{\ell}\sqrt{1+\xi^{*2}}\right]J_1\left(\frac{\alpha}{\ell}t^*\xi^*\right)J_1\left(\frac{\alpha}{\ell}\tau^*\xi^*\right)\mathrm{d}\xi^*. \tag{B.5}$$

By setting

$$\varphi(\alpha t^*) = \hat{\varphi}(t^*), \quad K(t^*,\tau^*) = \alpha K^*(t^*,\tau^*), \quad g(\alpha t^*) = \hat{g}(t^*)$$

$$K^*(t^*,\tau^*) = \tau^*\int_0^\infty \frac{1}{\xi^*}\left[1 - \frac{\ell'}{\ell}\sqrt{1+\xi^{*2}}\right]J_1\left(\frac{\alpha}{\ell}t^*\xi^*\right)J_1\left(\frac{\alpha}{\ell}\tau^*\xi^*\right)\mathrm{d}\xi^* \tag{B.6}$$

then eqn (B.4) takes the form

$$\hat{\varphi}(t^*) + \left(\frac{\alpha}{\ell}\right)^2\int_0^1 K^*(t^*,\tau^*)\hat{\varphi}(\tau^*)\,\mathrm{d}\tau^* = \hat{g}(t^*). \tag{B.7}$$

Then, by renaming the variables and for simplicity of notation

$$\hat{\varphi} \to \varphi, \quad \hat{g} \to g,$$
$$t^* \to t, \quad \tau^* \to \tau, \quad K^* \to K,$$
$$\lambda = \left(\frac{\alpha}{\ell}\right)^2, \quad \eta = \frac{\ell'}{\ell}, \quad \xi^* \to \xi; \tag{B.8}$$

we can rewrite eqn (B.7) in the following form

$$\varphi(t) + \lambda\int_0^1 K(t,\tau)\varphi(\tau)\,\mathrm{d}\tau = f(t) \tag{B.9}$$

with

$$K(t,\tau) = \tau\int_0^\infty \frac{1}{\xi}[1 - \eta\sqrt{1+\xi^2}]J_1(\sqrt{\lambda}t\xi)J_1(\sqrt{\lambda}\tau\xi)\,\mathrm{d}\xi. \tag{B.10}$$

By making the approximation

$$\frac{1}{\xi}(1-\eta\sqrt{1+\xi^2}) = \frac{1}{\xi}-\eta\sqrt{1+\frac{1}{\xi^2}} \approx \frac{1}{\xi}-\eta\left(1+\frac{1}{\xi}\right)=(1-\eta)\frac{1}{\xi}-\eta \tag{B.11}$$

the kernel becomes

$$K(t,\tau) \approx t\int_0^\infty \left[(1-\eta)\frac{1}{x}-\eta\right]J_1(\sqrt{\lambda}tx)J_1(\sqrt{\lambda}tx)\,\mathrm{d}x = (1-\eta)K_1(t,\tau)-\eta K_2(t,\tau) \tag{B.12}$$

with

$$K_1(t,\tau) = \tau\int_0^\infty \frac{1}{\xi}J_1(\sqrt{\lambda}t\xi)J_1(\sqrt{\lambda}\tau\xi)\,\mathrm{d}\xi, \quad K_2(t,\tau) = \tau\int_0^\infty J_1(\sqrt{\lambda}t\xi)J_1(\sqrt{\lambda}\tau\xi)\,\mathrm{d}\xi. \tag{B.13}$$

By recourse to the following formulae concerning the convergent integral (Watson, 1958)

$$I_1 = \int_0^\infty \xi^{-1}J_1(\sqrt{\lambda}t\xi)J_1(\sqrt{\lambda}\tau\xi)\,\mathrm{d}\xi = \begin{cases} \dfrac{1}{2}\dfrac{\tau}{t}, & \tau \leqslant t \\ \dfrac{1}{2}\dfrac{t}{\tau}, & \tau \geqslant t \end{cases} \tag{B.14}$$

and the divergent integral for $\tau = t$

$$I_2 = \int_0^\infty J_1(\sqrt{\lambda}t\xi)J_1(\sqrt{\lambda}\tau\xi)\,\mathrm{d}\xi = \begin{cases} \dfrac{1}{2}\dfrac{\tau}{\sqrt{\lambda}t^2}\,{}_2F_1\left(\dfrac{3}{2},\dfrac{1}{2};2;\left(\dfrac{\tau}{t}\right)^2\right), & \tau < t \\ \dfrac{1}{2}\dfrac{t}{\sqrt{\lambda}\tau^2}\,{}_2F_1\left(\dfrac{3}{2},\dfrac{1}{2};2;\left(\dfrac{t}{\tau}\right)^2\right), & \tau > t \end{cases} \tag{B.15}$$

the kernel takes the final form

$$K(t,\tau) \approx \begin{cases} \dfrac{1-\eta}{2}\dfrac{\tau^2}{t}-\dfrac{\eta}{2\sqrt{\lambda}}\left(\dfrac{\tau}{t}\right)^2{}_2F_1\left(\dfrac{3}{2},\dfrac{1}{2};2;\left(\dfrac{\tau}{t}\right)^2\right), & \tau < t \\ \dfrac{1-\eta}{2}t-\dfrac{\eta}{2\sqrt{\lambda}}\left(\dfrac{t}{\tau}\right)^2{}_2F_1\left(\dfrac{3}{2},\dfrac{1}{2};2;\left(\dfrac{t}{\tau}\right)^2\right), & \tau > t \end{cases}. \tag{B.16}$$

For $\eta \neq 0$ and $\tau \to t$ the kernel K_2 becomes unbounded. Indeed, an examination of eqn (B.16) reveals that

$$K_2(t,\tau) = \begin{cases} -\dfrac{1}{\pi}\left[\log\left(1-\dfrac{\tau^2}{t^2}\right)+O(1)\right], & \tau \to t-0 \\ -\dfrac{1}{\pi}\dfrac{\tau}{t}\left[\log\left(1-\dfrac{t^2}{\tau^2}\right)+O(1)\right], & \tau \to t+0, \end{cases} \tag{B.17}$$

i.e., the kernel K exhibits a logarithmic singularity at $\tau = t$. However, in the theory of Fredholm integral equations we can avoid the restrictive hypothesis of continuity (and consequently boundedness) of the kernel $K(t,\tau)$ by placing it in the Lebesgue class L^2 in the square $[0, 1] \times [0, 1]$ (Tricomi, 1957). Since from eqn (B.14) the kernel K_1 is bounded everywhere in $[0, 1] \times [0, 1]$ it only remains to show that the kernel K_2 is an L^2 function. The necessary and sufficient condition for the kernel $K_2(t,\tau)$ to belong in L^2 is

$$\|K_2\|^2 = \int_0^1\int_0^1 K_2^2(t,\tau)\,\mathrm{d}t\,\mathrm{d}\tau < \infty. \tag{B.18}$$

Considering first the limiting case $\tau \to t-0$, we get from the first of relationships (B.17)

$$|K_2(t,\tau)| = \frac{1}{\pi}\left|\log\left(1-\frac{\tau^2}{t^2}\right)+O(1)\right| \leqslant \frac{1}{\pi}\left\{\left|\log\left(1-\frac{\tau^2}{t^2}\right)\right|+|O(1)|\right\} < \frac{1}{\pi}\left|\log\left(1-\frac{\tau^2}{t^2}\right)\right|, \quad \tau \to t-0. \tag{B.19}$$

Substituting in eqn (B.18) we get

$$\|K_2(t,\tau)\|^2 < \frac{1}{\pi^2}\int_0^1 \mathrm{d}t\int_0^1\left(\log\left(1-\frac{\tau^2}{t^2}\right)\right)^2\mathrm{d}\tau, \quad \tau \to t-0. \tag{B.20}$$

By virtue of the transformation of variables

$$\xi = \frac{\tau}{t}$$

inequality (B.20) becomes

$$\|K_2\|^2 < \frac{1}{\pi^2}\int_0^1 t\,\mathrm{d}t \int_0^1 [\log(1-\xi^2)]^2\,\mathrm{d}\xi = \frac{1}{\pi^2}\int_0^1 t\,\mathrm{d}t\int_0^1 [\log(1-\xi)]^2\,\mathrm{d}\xi$$

$$+\frac{2}{\pi^2}\int_0^1 t\,\mathrm{d}t\int_0^1 \log(1-\xi)\log(1+\xi)\,\mathrm{d}\xi + \frac{1}{\pi^2}\int_0^1 t\,\mathrm{d}t\int_0^1 [\log(1+\xi)]^2\,\mathrm{d}\xi. \quad \text{(B.21)}$$

By making the substitutions

$$x = 1-\xi, \quad x' = 1+\xi$$

in the first two integrands and in the third integrand, respectively, and using also Schwarz's inequality

$$\int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \left[\int_a^b f^2(x)\,\mathrm{d}x\int_a^b g^2(x)\,\mathrm{d}x\right]^{1/2}$$

for the second integral of (B.21) we eventually find after the necessary manipulations

$$\|K_2\|^2 < 0.173 \quad \tau \to t-0. \quad \text{(B.22)}$$

Following a precisely similar piece of work one can also find from eqns (B.17) and (B.21)

$$\|K_2\|^2 < \frac{1}{\pi^2}\int_0^1 t\,\mathrm{d}t\int_0^1 \xi^2[\log(1-\xi^2)]^2\,\mathrm{d}\xi < \frac{1}{\pi^2}\int_0^1 t\,\mathrm{d}t\int_0^1 [\log(1-\xi^2)]^2\,\mathrm{d}\xi < 0.173 \quad \tau \to t+0 \quad \text{(B.23)}$$

which proves that the kernel K_2 is an L^2 function. Then, from eqns (B.16), (B.22) and (B.23) it is found

$$\|K(t,\tau)\| \approx \begin{cases} \left\|\dfrac{(1-\eta)}{2}\dfrac{\tau^2}{t} - \eta K_2(t,\tau)\right\| \leqslant \dfrac{(1-\eta)}{2}\left\|\dfrac{\tau^2}{t}\right\| + \eta\|K_2(t,\tau)\| < 0.5-0.1\eta = N & \tau < t \\ \left\|\dfrac{(1-\eta)}{2}t - \eta K_2(t,\tau)\right\| \leqslant \dfrac{(1-\eta)}{2}\|t\| + \eta\|K_2(t,\tau)\| < 0.5-0.1\eta = N & \tau > t \end{cases}. \quad \text{(B.24)}$$

Finally, it can be shown that the free term of the Fredholm integral equation

$$g(t) = \frac{1}{2}\frac{\sigma_0}{G}\frac{\ell'}{\ell^2}g_1(t) - \frac{\sigma_0}{G}g_2(t) \quad 0 \leqslant t \leqslant 1 \quad \text{(B.25)}$$

with

$$g_1(t) = t\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;t^2\right), \quad g_2(t) = t^{1/2}\delta(1-t) \quad \text{(B.26)}$$

belongs to L^1-class in [0, 1]. First,

$$|g_1(t)| = \frac{2}{\pi}|t^{-1}[\log(1-t^2)+O(1)]| < \frac{2}{\pi}|t^{-1}\log(1-t^2)|$$

from which it follows

$$\|g_1\|^2 = \int_0^1 g_1^2(t)\,\mathrm{d}t < \frac{2}{\pi}\int_0^1 t^{-2}[\log(1-t^2)]^2\,\mathrm{d}t.$$

By virtue of the following transformation of variables

$$x = 1-t^2$$

it turns out

$$\|g_1\|^2 < \frac{1}{\pi}\int_0^1 (1-x)^{-3/2}(\log x)^2\,\mathrm{d}x < \frac{1}{\pi}\int_0^1 (1-x)^{-2}(\log x)^2\,\mathrm{d}x$$

$$< \frac{1}{\pi}\int_0^\infty (1-x)^{-2}(\log x)^2\,\mathrm{d}x = \frac{2\pi}{3},$$

i.e. the function g_1 belongs to $L^2[0, 1]$ and consequently to $L^1[0, 1]$.

By virtue of the shifting property of the delta function it can be found

$$\|g_2\| = \int_0^1 t^{-1/2}\delta(1-t)\,\mathrm{d}t = 1$$

or in other words, g_2 belongs to $L^1[0, 1]$, but not to L^2 since

$$\|g_2\|^2 = \int_0^1 t^{-1}\delta(1-t)\delta(1-t)\,\mathrm{d}t = \delta(1-t).$$

The solution of the Fredholm integral equation for $\eta = 0$ following Liouville–Neumann method of successive substitutions provided that $\lambda < 1/N = 2$ is given by (Whittaker and Watson, 1948)

$$\varphi(t) = -\frac{\sigma_0}{G}t^{-1/2}\delta(1-t) - \lambda\varphi_1(t) + \lambda^2\varphi_2(t) - \cdots \tag{B.27}$$

where

$$\varphi_i(t) = \int_0^1 K_1(t,\tau)\varphi_{i-1}(\tau)\,\mathrm{d}\tau, \quad \varphi_0(t) = -\frac{\sigma_0}{G}t^{-1/2}\delta(1-t). \tag{B.28}$$

The above recursion formula leads to the following expressions

$$\varphi_1(t) = -\frac{\sigma_0}{2G}t, \quad \varphi_2(t) = -\frac{\sigma_0}{2G}\left[-\frac{1}{8}t^3 + \frac{1}{4}t\right], \quad \varphi_3(t) = -\frac{\sigma_0}{2G}\left[-\frac{1}{32}t^3 + \frac{3}{64}t + \frac{1}{192}t^5\right],$$

$$\varphi_4(t) = -\frac{\sigma_0}{2G}\left[-\frac{1}{9216}t^7 + \frac{1}{768}t^5 - \frac{3}{512}t^3 + \frac{19}{2304}t\right] \tag{B.29}$$

which are adequate for the convergency of the solution for $\lambda < 1/N = 2$. In Fig. B.1 it is demonstrated that the solution of the resolvent and Neumann's method for $\lambda = 1.5625$ coincide.

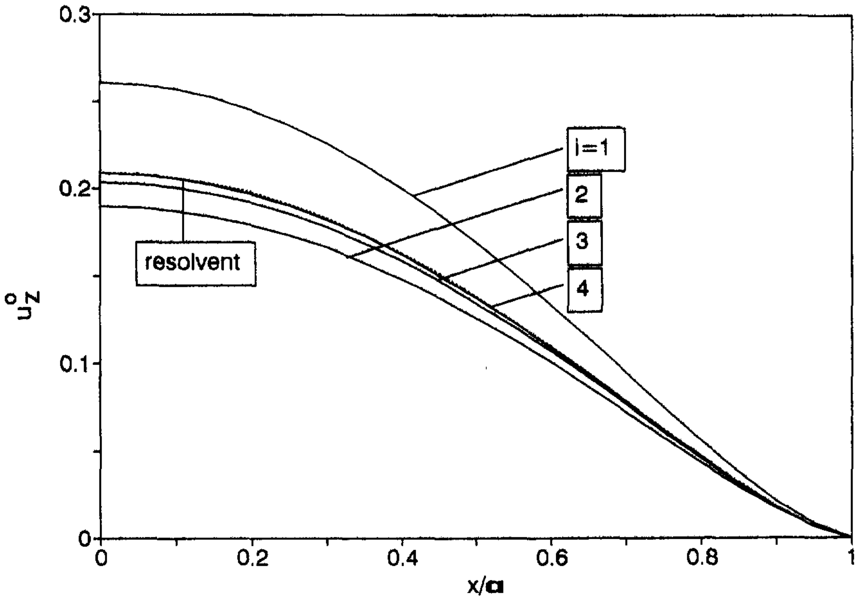

Fig. B.1. Convergency of solution by increasing the number of terms i in the truncated series of Newmann's method and comparison with resolvent method for $\lambda = 1.5625$.

APPENDIX C

Consider the improper integral expression (71)

$$u_z = \int_x^\alpha \varphi^*(t)\sqrt{t^2-x^2}\,\mathrm{d}t = \lim_{\eta\to 0}\left(\int_x^{\alpha-\eta} \varphi^*(t)\sqrt{t^2-x^2}\,\mathrm{d}t + \int_{\alpha-\eta}^{\alpha} \varphi^*(t)\sqrt{t^2-x^2}\,\mathrm{d}t\right) = I_1 + I_2 \tag{C.1}$$

where according to eqns (64) and (72) $\varphi^*(t)$ exhibits a logarithmic singularity at $t = \alpha$. By virtue of eqn (64)

$$\begin{aligned} |I_2| &= \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\alpha\left|\int_{\alpha-\eta}^{\alpha} -\frac{1}{\pi}\frac{1}{t}\left[\log\left(1-\frac{t^2}{\alpha^2}\right)+O(1)\right]\sqrt{t^2-x^2}\,\mathrm{d}t\right| \\ &< \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{\alpha}{\pi}\left|\int_{\alpha-\eta}^{\alpha} \frac{1}{t}\log\left(1-\frac{t^2}{\alpha^2}\right)\sqrt{t^2-x^2}\,\mathrm{d}t\right| \end{aligned} \tag{C.2}$$

and

$$\begin{aligned} |I_2| &< \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{\alpha}{\pi}\left|\int_{\alpha-\eta}^{\alpha} \frac{1}{t}\log\left(1-\frac{t^2}{\alpha^2}\right)\sqrt{\alpha^2\left(\frac{t^2}{\alpha^2}-\frac{x^2}{\alpha^2}\right)}\,\mathrm{d}t\right| < \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{\alpha}{\pi}\left|\int_{\alpha-\eta}^{\alpha} \log\left(1-\frac{t^2}{\alpha^2}\right)\mathrm{d}t\right| \\ &= \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{\alpha^2}{\pi}\left|\alpha\int_{1-\eta/\alpha}^{1} \log(1-\xi^2)\,\mathrm{d}\xi\right| = \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{\alpha\eta}{\pi}\left|-1.3+\log\frac{\eta}{\alpha}\right|. \end{aligned} \tag{C.3}$$

Next, consider the expression

$$\begin{aligned} u_z &= \int_x^{\alpha-\eta} \varphi^*(t)\sqrt{t^2-x^2}\,\mathrm{d}t + \varepsilon(\eta) \quad 0 \leqslant x < \alpha \\ \varepsilon(\eta) &= \frac{\sigma_0}{G}\frac{\ell'}{\ell^2}\frac{\alpha\eta}{\pi}\left|-1.3+\log\frac{\eta}{\alpha}\right| \to 0 \quad \text{as } \eta \to 0. \end{aligned} \tag{C.4}$$

Writing

$$u_z = \int_x^{\alpha-\eta} \hat{u}(t)\,\mathrm{d}v(t) + \varepsilon(\eta) \tag{C.5}$$

with

$$\hat{u}(t) = \frac{1}{3t}\varphi^*(t), \quad v(t) = (t^2-x^2)^{3/2} \tag{C.6}$$

we can carry out the integration by parts represented by

$$u_z = [\hat{u}(t)v(t)]_x^{\alpha-\eta} - \int_x^{\alpha-\eta} v(t)\,\mathrm{d}\hat{u}(t) + \varepsilon(\eta). \tag{C.7}$$

The above expression by virtue of (C.6) takes the form

$$u_z \approx \frac{1}{3\alpha}\varphi^*(\alpha-\eta)(\alpha^2-x^2)^{3/2} - I_1(x) + \varepsilon(\eta) \tag{C.8}$$

where

$$I_1(x) = \int_x^{\alpha-\eta} v(t)\,\mathrm{d}\hat{u}(t). \tag{C.9}$$

By using relations (C.6) the integral representation (C.9) becomes

$$I_1(x) = \int_x^{\alpha-\eta} (t^2-x^2)^{3/2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{3t}\varphi^*(t)\right\}\mathrm{d}t. \tag{C.10}$$

It is evident that the derivative appearing in the integrand is bounded for the range $x \leqslant t \leqslant \alpha-\eta$. Writing M for an upper bound on the absolute value we obtain

$$|I_1(x)| < M\int_x^\alpha (t^2-x^2)^{3/2}\,dt, \tag{C.11}$$

but since

$$(t^2-x^2)^{3/2} = (t+x)^{3/2}(t-x)^{3/2} \leqslant (2\alpha)^{3/2}(t-x)^{3/2} \tag{C.12}$$

we have further

$$|I_1(x)| < M(2\alpha)^{3/2}\int_x^\alpha (t-x)^{3/2}\,dt = \tfrac{2}{5}M(2\alpha)^{3/2}(\alpha-x)^{5/2}. \tag{C.13}$$

By setting

$$r = \alpha - x \tag{C.14}$$

it is evident from (C.8) and (C.13)

$$u_z(r)|_{r\to 0} = \frac{2\sqrt{2}}{3}\alpha^{1/2}\varphi^*(\alpha-\eta)r^{3/2} + O(r^{5/2}) + \varepsilon(\eta). \tag{C.15}$$